

THE POLYNOMIAL PROPERTY (V)

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ABSTRACT. Given Banach spaces E and F , we denote by $\mathcal{P}(^kE, F)$ the space of all k -homogeneous (continuous) polynomials from E into F , and by $\mathcal{P}_{\text{wb}}(^kE, F)$ the subspace of polynomials which are weak-to-norm continuous on bounded sets. It is shown that if E has an unconditional finite dimensional expansion of the identity, the following assertions are equivalent: (a) $\mathcal{P}(^kE, F) = \mathcal{P}_{\text{wb}}(^kE, F)$; (b) $\mathcal{P}_{\text{wb}}(^kE, F)$ contains no copy of c_0 ; (c) $\mathcal{P}(^kE, F)$ contains no copy of ℓ_∞ ; (d) $\mathcal{P}_{\text{wb}}(^kE, F)$ is complemented in $\mathcal{P}(^kE, F)$. This result was obtained by Kalton for linear operators. As an application, we show that if E has Pełczyński's property (V) and satisfies $\mathcal{P}(^kE) = \mathcal{P}_{\text{wb}}(^kE)$ then, for all F , every unconditionally converging $P \in \mathcal{P}(^kE, F)$ is weakly compact. If E has an unconditional finite dimensional expansion of the identity, then the converse is also true.

Given two Banach spaces E and F , we denote by $\mathcal{P}(^kE, F)$ the space of all k -homogeneous (continuous) polynomials from E into F , and by $\mathcal{P}_{\text{wb}}(^kE, F)$ the subspace of polynomials which are weak-to-norm continuous on bounded sets. This subspace has been studied by many authors: see, for instance, [3, 4, 17, 19]. Clearly, every polynomial in $\mathcal{P}_{\text{wb}}(^kE, F)$ takes bounded sets into relatively compact sets. Observe that $\mathcal{P}(^1E, F) = \mathcal{L}(E, F)$, the space of (linear bounded) operators from E into F , and that $\mathcal{P}_{\text{wb}}(^1E, F) = \mathcal{K}(E, F)$, the space of compact operators. The spaces $\mathcal{P}(^0E, F)$ and $\mathcal{P}_{\text{wb}}(^0E, F)$ may be identified with F .

Kalton studied in [23] the structure of the space $\mathcal{K}(E, F)$. In the present paper, we obtain versions of his results for the space $\mathcal{P}_{\text{wb}}(^kE, F)$, showing that $\mathcal{P}_{\text{wb}}(^kE, F)$ contains a copy of ℓ_∞ if and only if either E contains a complemented copy of ℓ_1 or F contains a copy of ℓ_∞ . We also prove that, for E having an unconditional finite dimensional expansion of the identity, the following assertions are equivalent: (a) $\mathcal{P}(^kE, F) = \mathcal{P}_{\text{wb}}(^kE, F)$; (b) $\mathcal{P}_{\text{wb}}(^kE, F)$ contains no copy of c_0 ; (c) $\mathcal{P}(^kE, F)$ contains no copy of ℓ_∞ ; (d) $\mathcal{P}_{\text{wb}}(^kE, F)$ is complemented in $\mathcal{P}(^kE, F)$.

As an application, we prove that, if E has property (V) (the definitions are given below) and $\mathcal{P}(^kE) = \mathcal{P}_{\text{wb}}(^kE)$, then every k -homogeneous unconditionally converging polynomial on E is weakly compact. If E has an unconditional finite dimensional expansion of the identity, then the converse is also true.

1991 *Mathematics Subject Classification.* Primary: 46B20; Secondary: 46E15.

Key words and phrases. Weakly continuous polynomial, unconditionally converging polynomial, weakly compact polynomial, weakly unconditionally Cauchy series, property (V).

The first named author was supported in part by DGICYT Grant PB 97-0349 (Spain).

The second named author was supported in part by DGICYT Grant PB 96-0607 (Spain).

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Throughout, E and F will denote Banach spaces, B_E is the closed unit ball of E and S_E is the unit sphere of E ; E^* will be the dual of E . The set of natural numbers is denoted by \mathbb{N} . As usual, (e_n) stands for the unit vector basis of c_0 .

A formal series $\sum x_n$ in E is *weakly unconditionally Cauchy* (w.u.C., for short) if, for every $\phi \in E^*$, we have $\sum |\phi(x_n)| < +\infty$. Equivalent definitions may be seen in [7, Theorem V.6]. The series is *unconditionally convergent* if every subseries converges. Equivalent definitions may be seen in [8, Theorem 1.9].

A polynomial $P \in \mathcal{P}(^k E, F)$ is *unconditionally converging* [15, 16] if, for each w.u.C. series $\sum x_n$ in E , the sequence $(P(\sum_{i=1}^n x_i))_n$ is convergent in F . The space of all unconditionally converging polynomials is denoted by $\mathcal{P}_{uc}(^k E, F)$. This class has been very useful for obtaining polynomial characterizations of Banach space properties (see [21]). We say that $P \in \mathcal{P}(^k E, F)$ is *(weakly) compact* if $P(B_E)$ is relatively (weakly) compact in F . Every weakly compact polynomial is unconditionally converging. For the general theory of polynomials on Banach spaces, we refer to [9, 26].

To each polynomial $P \in \mathcal{P}(^k E, F)$ we can associate a unique symmetric k -linear mapping $\hat{P} : E \times \binom{k}{\cdot} \times E \rightarrow F$ so that $P(x) = \hat{P}(x, \dots, x)$ and an operator $T_P : E \rightarrow \mathcal{P}(^{k-1} E, F)$ given by $T_P(x)(y) := \hat{P}(x, y, \binom{k-1}{\cdot}, y)$. It is well known that $P \in \mathcal{P}_{wb}(^k E, F)$ if and only if T_P is compact [3, Theorem 2.9].

Denote by $\mathcal{L}_s(E, \mathcal{P}(^{k-1} E, F))$ the space of all operators $C : E \rightarrow \mathcal{P}(^{k-1} E, F)$ such that

$$(C(x))^\wedge(y_1, \dots, y_{k-1}) = (C(y_1))^\wedge(x, y_2, \dots, y_{k-1}) \quad (x, y_1, \dots, y_{k-1} \in E),$$

where $(C(x))^\wedge$ stands for the symmetric $(k-1)$ -linear mapping associated to $C(x)$.

Proposition 1. *The mapping $T : \mathcal{P}(^k E, F) \rightarrow \mathcal{L}_s(E, \mathcal{P}(^{k-1} E, F))$ given by $T(P) = T_P$ is a surjective linear isomorphism.*

Proof. Clearly, T is well defined, linear and injective. Since

$$\|T_P\| = \sup_{x \in B_E} \|T_P(x)\| = \sup_{x, y \in B_E} \|\hat{P}(x, y, \binom{k-1}{\cdot}, y)\| \leq \|\hat{P}\| \leq \frac{k^k}{k!} \|P\|$$

[26, Theorem 2.2], we have that T is continuous. To see that it is surjective, take $C \in \mathcal{L}_s(E, \mathcal{P}(^{k-1} E, F))$, and define $A : E \times \binom{k}{\cdot} \times E \rightarrow F$ by $A(y_1, \dots, y_k) := (C(y_1))^\wedge(y_2, \dots, y_k)$, and $P(x) = A(x, \binom{k}{\cdot}, x)$. Then

$$T_P(x)(y) = A(x, y, \binom{k-1}{\cdot}, y) = C(x)(y) \quad (x, y \in E).$$

Hence, $T_P = C$. □

The subspace of all operators in $\mathcal{L}_s(E, \mathcal{P}(^{k-1} E, F))$ which are compact (resp. weakly compact) will be denoted by $\mathcal{K}_s(E, \mathcal{P}(^{k-1} E, F))$ (resp. $\mathcal{W}_s(E, \mathcal{P}(^{k-1} E, F))$). Given $C \in \mathcal{K}_s(E, \mathcal{P}(^{k-1} E, F))$, the symmetry of C easily implies that $C(E) \subseteq \mathcal{P}_{wb}(^{k-1} E, F)$.

The proof of [5, Proposition 5.3] gives:

Proposition 2. *For $n \geq m$, the space $\mathcal{P}(^m E, F)$ (resp. $\mathcal{P}_{wb}(^m E, F)$) is isomorphic to a complemented subspace of $\mathcal{P}(^n E, F)$ (resp. $\mathcal{P}_{wb}(^n E, F)$).*

Theorem 3. *The space $\mathcal{P}_{\text{wb}}(^kE, F)$ contains a copy of ℓ_∞ if and only if either F contains a copy of ℓ_∞ or E contains a complemented copy of ℓ_1 .*

Proof. If $\mathcal{P}_{\text{wb}}(^kE, F)$ contains a copy of ℓ_∞ , a fortiori the space $\mathcal{K}(E, \mathcal{P}_{\text{wb}}(^{k-1}E, F))$ contains it. Therefore [23, Theorem 4], either E contains a complemented copy of ℓ_1 or $\mathcal{P}_{\text{wb}}(^{k-1}E, F)$ contains a copy of ℓ_∞ . Repeating the process, we conclude that either E contains a complemented copy of ℓ_1 or $\mathcal{P}_{\text{wb}}(^0E, F) \equiv F$ contains a copy of ℓ_∞ .

Conversely, if F contains a copy of ℓ_∞ , since $F \equiv \mathcal{P}_{\text{wb}}(^0E, F)$ is isomorphic to a subspace of $\mathcal{P}_{\text{wb}}(^kE, F)$ (Proposition 2), we obtain that $\mathcal{P}_{\text{wb}}(^kE, F)$ contains a copy of ℓ_∞ . If E contains a complemented copy of ℓ_1 , then E^* contains a copy of ℓ_∞ ; since $E^* = \mathcal{P}_{\text{wb}}(^1E)$ is isomorphic to a subspace of $\mathcal{P}_{\text{wb}}(^1E, F)$, which is in turn isomorphic to a subspace of $\mathcal{P}_{\text{wb}}(^kE, F)$, we obtain that the latter contains a copy of ℓ_∞ . \square

The proof of [23, Lemma 2] yields:

Lemma 4. *Assume E is separable, $\mathcal{P}_{\text{wb}}(^kE, F)$ is complemented in $\mathcal{P}(^kE, F)$, and an operator $\Phi : \ell_\infty \rightarrow \mathcal{P}(^kE, F)$ is given with the following properties:*

- (a) $\Phi(e_n) \in \mathcal{P}_{\text{wb}}(^kE, F)$ for all n ;
- (b) the subset $\{\Phi(\xi)(x) : \xi \in \ell_\infty, x \in E\} \subset F$ is separable.

Then, for every infinite subset $M \subseteq \mathbb{N}$, there exists an infinite subset $M_0 \subseteq M$ with $\Phi(\xi) \in \mathcal{P}_{\text{wb}}(^kE, F)$ for all $\xi \in \ell_\infty(M_0)$.

Lemma 5. *Suppose E contains a complemented copy of ℓ_1 . Then $\mathcal{P}_{\text{wb}}(^kE, F)$ is uncomplemented in $\mathcal{P}(^kE, F)$ for every F and $k \in \mathbb{N}$ ($k > 1$).*

Proof. As in [23, Lemma 3], we can reduce the problem to the case $E = \ell_1$.

Fix $v \in S_F$ and define the operator

$$\Phi : \ell_\infty \longrightarrow \mathcal{P}(^k\ell_1, F)$$

by

$$\Phi(\xi)(x) = \sum_{i=1}^{\infty} \xi_i x_i^k v \quad \text{for } \xi = (\xi_i)_{i=1}^{\infty} \in \ell_\infty \text{ and } x = (x_i)_{i=1}^{\infty} \in \ell_1.$$

Since

$$\|\Phi(\xi)\| = \sup_{x \in B_{\ell_1}} \left\| \sum_{i=1}^{\infty} \xi_i x_i^k v \right\| \leq \|\xi\| \cdot \sup_{x \in B_{\ell_1}} \sum_{i=1}^{\infty} |x_i^k| = \|\xi\|,$$

Φ is continuous (easily, it is even an isometric embedding).

We claim that $\Phi(\xi) \in \mathcal{P}_{\text{wb}}(^k\ell_1, F)$ if and only if $\xi \in c_0$. Indeed, let

$$T_{\Phi(\xi)} : \ell_1 \longrightarrow \mathcal{P}(^{k-1}\ell_1, F)$$

be the associated operator given by

$$T_{\Phi(\xi)}(x)(y) = \sum_{i=1}^{\infty} \xi_i x_i y_i^{k-1} v \quad \text{for } x = (x_i), y = (y_i) \in \ell_1.$$

Since, for $n \neq m$,

$$\begin{aligned} \|(T_{\Phi(\xi)}(e_n) - T_{\Phi(\xi)}(e_m))(y)\| &= \\ \|T_{\Phi(\xi)}(e_n - e_m)(y)\| &= \left\| \sum_{i=1}^{\infty} \xi_i (\delta_{in} - \delta_{im}) y_i^{k-1} v \right\| = |\xi_n y_n^{k-1} - \xi_m y_m^{k-1}|, \end{aligned}$$

we have

$$\max\{|\xi_n|, |\xi_m|\} \leq \|T_{\Phi(\xi)}(e_n) - T_{\Phi(\xi)}(e_m)\| \leq |\xi_n| + |\xi_m|,$$

it follows that $T_{\Phi(\xi)}$ is compact if and only if $\xi \in c_0$ (see [7, Exercise VII.5]).

By Lemma 4, there is an infinite $M \subseteq \mathbb{N}$ such that $\Phi(\xi) \in \mathcal{P}_{\text{wb}}({}^k\ell_1, F)$ for all $\xi \in \ell_\infty(M)$, which contradicts the above claim. \square

In the linear case ($k = 1$), Kalton needs to assume that F is infinite dimensional for the validity of Lemma 5 [23, Lemma 3]. Taking $k > 1$, we can drop this condition. Observe that, if $k = 1$ and $\dim(F) < \infty$, we have

$$\mathcal{P}_{\text{wb}}({}^1E, F) = \mathcal{K}(E, F) = \mathcal{L}(E, F) = \mathcal{P}({}^1E, F),$$

so the conclusion of the Lemma is not true.

An *unconditional finite dimensional expansion of the identity* for a Banach space E is a sequence of finite dimensional operators $A_n : E \rightarrow E$ such that for each $x \in E$,

$$x = \sum_{n=1}^{\infty} A_n(x)$$

unconditionally. This condition is a bit more general than having an unconditional basis [22].

Lemma 6. *Suppose E has an unconditional finite dimensional expansion of the identity and let $P \in \mathcal{P}({}^kE, F)$. Then there is a w.u.C. series $\sum P_i$ in $\mathcal{P}_{\text{wb}}({}^kE, F)$ such that, for all $x \in E$, $P(x) = \sum_{m=1}^{\infty} P_m(x)$ unconditionally.*

Proof. There is a sequence $(A_i) \subset \mathcal{K}(E, E)$ such that, for every $x \in E$, we have $x = \sum_{i=1}^{\infty} A_i(x)$ unconditionally. Then,

$$\begin{aligned} P\left(\sum_{i=1}^n A_i(x)\right) &= \sum_{i_1, \dots, i_k=1}^n \hat{P}(A_{i_1}(x), \dots, A_{i_k}(x)) \\ &= \sum_{m=1}^n \left(\sum_{\max\{i_1, \dots, i_k\}=m} \hat{P}(A_{i_1}(x), \dots, A_{i_k}(x)) \right) \\ &= \sum_{m=1}^n P_m(x) \end{aligned}$$

where $P_m \in \mathcal{P}_{\text{wb}}({}^kE, F)$.

Choosing finite subsets I_1, \dots, I_k of integers, we have

$$\begin{aligned} \left\| \sum_{i_1 \in I_1, \dots, i_k \in I_k} \hat{P}(A_{i_1}(x), \dots, A_{i_k}(x)) \right\| &= \left\| \hat{P} \left(\sum_{i_1 \in I_1} A_{i_1}(x), \dots, \sum_{i_k \in I_k} A_{i_k}(x) \right) \right\| \\ &\leq \|\hat{P}\| \cdot \left\| \sum_{i_1 \in I_1} A_{i_1}(x) \right\| \cdot \dots \cdot \left\| \sum_{i_k \in I_k} A_{i_k}(x) \right\|. \end{aligned}$$

Hence, the series

$$\sum_{i_1, \dots, i_k=1}^{\infty} \hat{P}(A_{i_1}(x), \dots, A_{i_k}(x))$$

is unconditionally convergent for all $x \in E$ [8, Theorem 1.9]. Therefore, $P(x) = \sum_{m=1}^{\infty} P_m(x)$ unconditionally.

Moreover, by the uniform boundedness principle [26, Theorem 2.6], we have

$$\sup_{I \subset \mathbb{N} \text{ finite}} \left\| \sum_{i \in I} P_i \right\| < \infty.$$

So $\sum P_i$ is w.u.C. in $\mathcal{P}_{\text{wb}}(^kE, F)$ [7, Theorem V.6]. \square

Theorem 7. *Suppose E has an unconditional finite dimensional expansion of the identity and let $k \in \mathbb{N}$ ($k > 1$). Then the following assertions are equivalent:*

- (a) $\mathcal{P}(^kE, F) = \mathcal{P}_{\text{wb}}(^kE, F)$;
- (b) $\mathcal{P}_{\text{wb}}(^kE, F)$ contains no copy of c_0 ;
- (c) $\mathcal{P}(^kE, F)$ contains no copy of ℓ_{∞} ;
- (d) $\mathcal{P}_{\text{wb}}(^kE, F)$ is complemented in $\mathcal{P}(^kE, F)$.

Proof. (a) \Rightarrow (c): Assume $\mathcal{P}(^kE, F) = \mathcal{P}_{\text{wb}}(^kE, F)$ contains a copy of ℓ_{∞} . By Theorem 3, either E contains a complemented copy of ℓ_1 or F contains a copy of ℓ_{∞} . Lemma 5 implies that E contains no complemented copy of ℓ_1 , so F contains a copy of ℓ_{∞} . Take a normalized basic sequence $(x_i) \subset E$ and a bounded sequence of coefficient functionals $(\phi_n) \subset E^*$ ($\phi_i(x_j) = \delta_{ij}$). Define $P \in \mathcal{P}(^kE, \ell_{\infty})$ by $P(x) := (\phi_n(x)^k)_n$. Then, for $i \neq j$, we have $\|P(x_i) - P(x_j)\| \geq |\phi_i(x_i)^k - \phi_i(x_j)^k| = 1$. Hence, denoting by $J : \ell_{\infty} \rightarrow F$ an isomorphism, we get $J \circ P \in \mathcal{P}(^kE, F) \setminus \mathcal{P}_{\text{wb}}(^kE, F)$, a contradiction.

(c) \Rightarrow (b): The proof is the same of the linear case [23, Theorem 6] with slight modifications.

(b) \Rightarrow (a): Take $P \in \mathcal{P}(^kE, F)$. Consider the w.u.C. series $\sum P_i$ given by Lemma 6 with $P_i \in \mathcal{P}_{\text{wb}}(^kE, F)$. Since this space contains no copy of c_0 , the series is unconditionally convergent [7, Theorem V.8]. Clearly, its sum must be P . Since the space $\mathcal{P}_{\text{wb}}(^kE, F)$ is closed, we conclude that $P \in \mathcal{P}_{\text{wb}}(^kE, F)$.

(a) \Rightarrow (d) is trivial.

(d) \Rightarrow (a): If $\mathcal{P}_{\text{wb}}(^kE, F)$ is complemented in $\mathcal{P}(^kE, F)$, Lemma 5 implies that E contains no complemented copy of ℓ_1 . Suppose there is $P \in \mathcal{P}(^kE, F) \setminus \mathcal{P}_{\text{wb}}(^kE, F)$. By Lemma 6, we can find a sequence $(P_i) \subset \mathcal{P}_{\text{wb}}(^kE, F)$ so that $P(x) = \sum_{i=1}^{\infty} P_i(x)$ unconditionally for each $x \in E$, and $\sum P_i$ is w.u.C. but is not unconditionally

convergent since $P \notin \mathcal{P}_{\text{wb}}(^kE, F)$. Hence, we can find $\epsilon > 0$, and an increasing sequence (m_j) of integers such that, for each j , the polynomial

$$C_j := \sum_{i=m_j+1}^{m_{j+1}} P_i$$

satisfies $\|C_j\| > \epsilon$.

Define now $\Phi : \ell_\infty \rightarrow \mathcal{P}(^kE, F)$ by $\Phi(\xi)(x) = \sum \xi_j C_j(x)$ for $\xi = (\xi_j)_{j=1}^\infty \in \ell_\infty$ and $x \in E$. Since the series $\sum P_i(x)$ is unconditionally convergent, so is the series $\sum \xi_j C_j(x)$. The set $\{\Phi(\xi)(x) : \xi \in \ell_\infty, x \in E\}$ is contained in the closed linear span of $\{C_j(E) : j \in \mathbb{N}\}$ which is separable by the compactness of C_j . On the other hand, $\Phi(e_j) = C_j \in \mathcal{P}_{\text{wb}}(^kE, F)$ for all j . By Lemma 4, there is an infinite subset $M \subset \mathbb{N}$ such that $\Phi(\xi) \in \mathcal{P}_{\text{wb}}(^kE, F)$ for each $\xi \in \ell_\infty(M)$.

Therefore, for each $\xi \in \ell_\infty(M)$, the series $\sum \xi_j C_j$ is weak subseries convergent. The Orlicz-Pettis theorem then implies that it is unconditionally convergent. In particular,

$$\lim_{\substack{j \in M \\ j \rightarrow \infty}} \|C_j\| = 0,$$

a contradiction. \square

Observe that the unconditional finite dimensional expansion of the identity is used only in (b) \Rightarrow (a) and (d) \Rightarrow (a).

In the linear case ($k = 1$), the restriction $\dim(F) = \infty$ is required for the validity of Theorem 7 [23, Theorem 6].

REMARK 8. In order to highlight the difference between Theorem 3 and Theorem 7, let us consider the spaces $\mathcal{P}(^k\ell_p, \ell_q)$ and $\mathcal{P}_{\text{wb}}(^k\ell_p, \ell_q)$ for $1 < p, q < \infty$ and $k > 1$.

If $kq > p$, then the space $\mathcal{P}(^k\ell_p, \ell_q)$ contains a copy of ℓ_∞ . Indeed, define $J : \ell_\infty \rightarrow \mathcal{P}(^k\ell_p, \ell_q)$ by

$$J(\xi)(x) = (\xi_i x_i^k)_{i=1}^\infty.$$

Since, for all n ,

$$|\xi_n| = \|J(\xi)(e_n)\| \leq \|J(\xi)\| = \sup_{x \in B_{\ell_p}} \|J(\xi)(x)\| \leq \|\xi\|,$$

we get that J is an isometric embedding.

However, the space $\mathcal{P}_{\text{wb}}(^k\ell_p, \ell_q)$ contains no copy of ℓ_∞ since it is separable. In fact, it is the norm closure of the space of all finite type polynomials generated by the mappings of the form $\phi^n \otimes y$, for $\phi \in (\ell_p)^*$, $y \in \ell_q$ [4, Proposition 2.7].

On the other hand, if $\xi \in c_0$, then $J(\xi)$ is in the closure of the finite type polynomials, so $J(\xi) \in \mathcal{P}_{\text{wb}}(^k\ell_p, \ell_q)$. Hence, this space contains $J(c_0)$, an isometric copy of c_0 .

Recall moreover that, if $kq < p$, then the space $\mathcal{P}(^k\ell_p, \ell_q)$ is reflexive [1, 4.3].

Every polynomial $P \in \mathcal{P}(^kE, F)$ has an extension, called *the Aron-Berner extension*, to a polynomial $\overline{P} \in \mathcal{P}(^kE^{**}, F^{**})$ (see [2, 18, 21]). If $P \in \mathcal{P}_{\text{wb}}(^kE)$, then $\overline{P} \in \mathcal{P}(^kE^{**})$ is weak-star continuous on bounded sets [25].

Recall that a Banach space has *property (V)*, introduced in [27], if every unconditionally converging operator on E is weakly compact. Every $C(K)$ space has property (V) [27].

We shall need the following result:

Theorem 9. [21] *The following assertions are equivalent:*

- (a) *The space E has property (V);*
- (b) *for all F and $k \in \mathbb{N}$, the Aron-Berner extension of every $P \in \mathcal{P}_{\text{uc}}(^kE, F)$ is F -valued;*
- (c) *There is $k \in \mathbb{N}$ such that, for all F , the Aron-Berner extension of every $P \in \mathcal{P}_{\text{uc}}(^kE, F)$ is F -valued.*

Easily, if a polynomial is weakly compact, then its Aron-Berner extension is F -valued [6].

Theorem 10. *For $k \in \mathbb{N}$, consider the assertions:*

- (a) *E has property (V) and $\mathcal{P}(^kE) = \mathcal{P}_{\text{wb}}(^kE)$;*
- (b) *for each F , every $P \in \mathcal{P}_{\text{uc}}(^kE, F)$ is weakly compact.*

Then (a) \Rightarrow (b). If, moreover, E has an unconditional finite dimensional expansion of the identity, then (b) \Rightarrow (a).

Proof. (a) \Rightarrow (b): Given $P \in \mathcal{P}_{\text{uc}}(^kE, F)$, by Theorem 9, the range of its Aron-Berner extension \overline{P} is contained in F . Take a net $(x_\alpha) \subset B_E$. We can assume that (x_α) is weak Cauchy and so it converges in the weak-star topology to some $z \in E^{**}$.

Let $\psi \in F^*$. Then $\psi \circ P \in \mathcal{P}_{\text{wb}}(^kE)$ and so,

$$\psi \circ P(x_\alpha) = \psi \circ \overline{P}(x_\alpha) = \overline{\psi \circ P}(x_\alpha) \longrightarrow \overline{\psi \circ P}(z) = \psi \circ \overline{P}(z).$$

Therefore, the net $(P(x_\alpha))$ is weakly convergent to $\overline{P}(z) \in F$. So, $P(B_E)$ is relatively weakly compact.

(b) \Rightarrow (a): By the comment preceding this Theorem, and by Theorem 9, (b) implies that E has property (V). Also, every polynomial in $\mathcal{P}(^kE, \ell_1)$ is compact. From this, we obtain that $\mathcal{P}(^kE)$ contains no copy of c_0 [10, Corollary 8]. A fortiori, $\mathcal{P}_{\text{wb}}(^kE)$ contains no copy of c_0 . Since E has an unconditional finite dimensional expansion of the identity, we conclude from Theorem 7 that $\mathcal{P}_{\text{wb}}(^kE) = \mathcal{P}(^kE)$. \square

We do not know if the condition on the existence of an unconditional finite dimensional expansion of the identity may be removed from Theorem 10. In fact, if (b) is satisfied, since $\mathcal{P}(^{k-1}E)$ contains no copy of c_0 and E has property (V), we have

$$\begin{aligned} \mathcal{P}(^kE) &= \mathcal{L}_s(E, \mathcal{P}(^{k-1}E)) = \mathcal{W}_s(E, \mathcal{P}(^{k-1}E)), \\ \mathcal{P}_{\text{wb}}(^kE) &= \mathcal{K}_s(E, \mathcal{P}(^{k-1}E)). \end{aligned}$$

So, we only have to show that $\mathcal{K}_s(E, \mathcal{P}(^{k-1}E)) = \mathcal{W}_s(E, \mathcal{P}(^{k-1}E))$. There are many conditions on E (apart from the existence of an unconditional finite dimensional expansion of the identity) that imply this equality [13, 14], and it is not known if there are Banach spaces X, Y such that $\mathcal{K}(X, Y) \neq \mathcal{W}(X, Y)$ while $\mathcal{K}(X, Y)$ contains no copy of c_0 [11].

Recall that the condition $\mathcal{P}({}^kE) = \mathcal{P}_{\text{wb}}({}^kE)$ implies that E contains no copy of ℓ_1 [20].

It is proved in [15] that if $\mathcal{P}({}^kE) = \mathcal{P}_{\text{wb}}({}^kE)$ and E has property (u) then, for all F , every $P \in \mathcal{P}_{\text{uc}}({}^kE, F)$ is weakly compact. Part (a) \Rightarrow (b) of Theorem 10 is stronger than the result of [15]. The latter cannot be applied, for instance, to the space $c_0 \hat{\otimes}_{\pi} c_0$ which fails property (u) [24] while it does have property (V) [12]. For the definition of property (u) and its relationship to property (V), see [27]. Recall in particular that, if E has property (u) and contains no copy of ℓ_1 , then E has property (V) [27, Proposition 2].

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